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On Finite Amplitude Roll Cell Disturbances in a Fluid Layer Subjected to Heat and Mass Transfer

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The system consists of a thin, transversely infinite horizontal fluid layer initially in mechanical but not thermostatic or speciestatic equilibrium. The top and bottom surfaces of the layer are assumed to be unconstrained, and the stability of the system to buoyancy driven finite amplitude roll cell disturbances is investigated. The system is capable of exhibiting both a subcritical stationary and oscillatory instability. The average vertical transport of heat and mass is computed and it is found that the convective transport associated with a stationary type of instability is much greater than that corresponding to an oscillatory instability.

In engineering, meteorology, biology, and other branches of science, many of the complex problems which are encountered involve the interaction of transport and transformation processes which are coupled at the macroscopic level. In many such dynamic systems the stability of the system is the object of prime interest. Analytical treatments of such systems have proceeded at a slow rate owing to the inherent complexity of a reasonable mathematical description. Moreover, most of the available stability analyses deal with the stability of the system to infinitesimal disturbances and utilize the simplification of a linearized mathematical model. The stability of the system to infinitesimal disturbances may, or may not, provide an adequate description of the stability of the system to disturbances of finite amplitude (see reference 9). It does not in the system to be considered here.

The present analysis deals with the convective instability of a system undergoing heat and mass transfer to one form of finite amplitude disturbance. The distortion of the mean temperature and concentration fields by the disturbance, the configuration of the local temperature and concentration fields, and the transport of heat and mass at a finite amplitude roll cell equilibrium state are considered; the prediction of the cellular pattern with the greatest tendency to form in the physical system and the tendency for a change in cell size are not considered. Not only does the analysis help to fill a gap in one area of convective instability in general, but at the same time it illustrates such features as a subcritical instability, a bifurcation point where the regime of the developed instability undergoes a transition from a stationary instability to an oscillatory instability, and the poor convective transport characteristics of an oscillatory instability.

DESCRIPTION OF THE SYSTEM

The system to be investigated is a transversely infinite horizontal fluid layer initially in mechanical but not thermostatic or speciestatic equilibrium, with heat and mass transfer across the layer and concomitantly a potentially unstable density gradient. The only force presumed to exist in the initial quiescent system is the force of gravity g, which is constant to very good approximation; consequently, the condition for mechanical equilibrium is

$$(\nabla \rho) \times g = 0 \tag{2.1}$$

By thermostatics $\rho = f(\theta^{\bullet}, \Lambda^{\bullet}, p^{\bullet})$ and a truncated Taylor expansion leads to

$$\rho = \rho_r \{1 + \gamma(\theta^{\bullet} - \theta_r^{\bullet}) + \beta(\Lambda^{\bullet} - \Lambda_r^{\bullet})\} \tag{2.2}$$

Here the first-order pressure variation of the density is assumed to be negligible. Substitution of Equation (2.2) into (2.1) leads to the equilibrium criterion, which is appropriate for the present system.

$$(\gamma \nabla \theta^{\bullet} + \beta \nabla \Lambda^{\bullet}) \times \mathbf{g} = 0 \tag{2.3}$$

This criterion restricts the temperature and concentration gradients in the initial quiescent system. The present analysis is restricted to the case in which the configuration of the initial state of the system is such that the temperature and concentration gradients are antiparallel to the gravitational field and $\gamma(\theta_B^{\bullet} - \theta_T^{\bullet}) < 0$, $\beta(\Lambda_B^{\bullet} - \Lambda_T^{\bullet}) > 0$, where $(\theta_B^{\bullet} - \theta_T^{\bullet})$ and $(\Lambda_B^{\bullet} - \Lambda_T^{\bullet})$ are the temperature and concentration differences across the quiescent system. Thus, the temperature field tends to destabilize and the concentration field tends to stabilize the potentially unstable quiescent state of the system. Other configurations of the quiescent state of the system have been discussed by Sani (*5*). The present analysis is subjected to the above restrictions, since such a system can exhibit an instability with a monotone, as well as an oscillatory, temporal behavior. Consequently, the finite amplitude behavior of the system in both cases can be investigated and compared with a minimum of additional labor.

When a unitless temperature gradient, the thermal Rayleigh number, $R_t^{\bullet}$

$$R_t^{\bullet} \equiv \frac{l^3\, g \Delta\rho_t N_{Pr}}{\nu^2 \rho_r} \tag{2.4}$$

exceeds a certain critical value, a flow is established, that is, an instability is realized. In the case of a transversely infinite layer of fluid subjected to heat transfer only, the instability leads to the establishment of highly ordered convection cells (see reference *1*, p. 11) called after Bénard. It is expected that the present system will also exhibit a highly ordered array of convection cells at the point of instability.

MATHEMATICAL FORMULATION

When the Boussinesq approximations (*8*) are utilized and when consideration is focused on two-dimensional rolls $\left(v = \frac{\partial}{\partial y} = 0\right)$, the unitless mathematical representation of the dynamic state of a Newtonian fluid within the layer assumes the form

$$u_x + w_z = 0, \tag{3.1}$$

$$\Upsilon_t - \Upsilon_{zz} = -\,(\overline{wT})_z, \tag{3.2}$$

$$\Gamma_t - \eta\, \Gamma_{zz} = -\,(\overline{wC})_z, \tag{3.3}$$

$$T_t - \nabla^2 T = -\,w\Upsilon_z - uT_x - wT_z + (\overline{wT})_z, \tag{3.4}$$

$$C_t - \eta\, \nabla^2 C = -\,w\Gamma_z - uC_x - wC_z + (\overline{wC})_z, \tag{3.5}$$

$$\left(\frac{\partial}{\partial t} - N_{Pr}\nabla^2\right)\nabla^2 w - N_{Pr}T_{xx} + N_{Sc}C_{xx} =$$

$$(uu_x + wu_z)_{xz} - (uw_x + ww_z)_{xx} \tag{3.6}$$

An overbar denotes an average in x, that is

$$\overline{Q} \equiv \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} Q(x, z, t)\, dx$$

and the dimensionless temperature field θ and dimensionless concentration field Λ have been decomposed into mean (Υ, Γ) and perturbation (T, C) parts. Namely

$$\theta = \Upsilon + T \quad , \quad \Lambda = \Gamma + C$$

where here

$$\overline{\Upsilon} = \Upsilon, \quad \overline{\Gamma} = \Gamma, \quad \overline{T} = 0, \quad \overline{C} = 0$$

It is instructive at this point to mention the major approximations incorporated in Equations (3.1) to (3.6): (1) All physical properties, except the density appearing in the body force term, are assumed to be constant. (2) The energy generated through viscous dissipation is neglected. (3) The density is a linear function of temperature and concentration. (4) The concentration and the flux of the diffusing species are small. Some results of Spiegel and Veronis (*8*) appear to indicate that these approximations are valid if the fluid layer is thin and if the gradients of concentration and temperature across the layer are sufficiently small.

Now the boundary conditions are specified in order to complete the mathematical characterization of the system. The fluid layer is bounded by the planes $z = 0$ and $z = 1$, which are normal to the gravitational field. On these planes the normal velocity, viscous traction (assuming the viscous traction exerted on the boundary by the surrounding medium is negligible), and the temperature and concentration perturbations, that is, T and C, are presumed to vanish. That is, the planes $z = 0$ and $z = 1$ are so-called conducting-free surfaces. (The effects of surface tension forces and deflection of the free surfaces are neglected.) The mathematical characterization of this type of boundary is

$$C = T = w = w_{zz} = 0 \quad \text{at} \quad z = 0 \text{ and } z = 1 \tag{3.7}$$

Although a conducting-free surface, as defined here, characterizes a rather unrealistic communication of the system with its surroundings, it results in computational simplicity; however, more realistic boundary conditions can be treated by the method used here.

If the specification of the physical system is such that the top surface $z = 1$ is maintained at temperature θ_T and concentration Λ_T, and the bottom surface $z = 0$ is maintained at temperature $\theta_B > \theta_T$ $(\theta_B < \theta_T)$ and concentration $\Lambda_B > \Lambda_T$ $(\Lambda_B < \Lambda_T)$ for $\gamma < 0$, $\beta > 0$ (for $\gamma > 0$, $\beta < 0$), the mathematical description of the system (3.1) to (3.7) has the following time-independent solution:

$$u = w = T = C = 0, \quad \Upsilon_o = -\,R_t^{\bullet}z + \Upsilon_B,$$
$$\Gamma_o = -\,\eta^2 R_c^{\bullet} z + \Gamma_B \tag{3.8}$$

$$R_c^{\bullet} \equiv \frac{l^3 g \Delta\rho_c N_{Sc}}{\nu^2 \rho_r} \tag{3.9}$$

The time-independent solution (3.8) represents the initial quiescent state of the physical system. This state is perturbed in order to determine the convective stability of the system; that is, the existence and location of the smallest branch point of the nonlinear system (3.1) to (3.7) is investigated. In order to complete the boundary conditions of the perturbed system the following boundary conditions are appended to those appearing in (3.7).

$$\Upsilon - \Upsilon_o = 0, \quad \Gamma - \Gamma_o = 0 \quad \text{at} \quad z = 0 \text{ and } z = 1 \tag{3.10}$$

After the method of Stuart and Watson (*10, 12*), the following harmonic analysis is made of the perturbations:

$$q = \frac{1}{2}\sum_{n=1}^{\infty} \{q_n(z, t) \exp\,(i\alpha n\pi x) +$$
$$\tilde{q}_n(z, t) \exp\,(-i\alpha n\pi x)\} \tag{3.11}$$

where

$$q_n(z, t) = A^n(t)\left\{q_{no}(z) + \sum_{m=1}^{\infty} |A|^{2m}\, q_{nm}(z)\right\} \tag{3.12}$$

$|A| \equiv$ modulus A, an over $\sim$ denotes a complex conjugate and q stands for w, T, or C. The series representation for u follows from the representation for w and the continuity Equation (3.1), for example

$$u = \frac{i}{2}(\alpha\pi)^{-1} \sum_{n=1}^{\infty} n^{-1} \{w_{n,z} \exp(i\alpha n\pi x) - \tilde{w}_{n,z} \exp(-i\alpha n\pi x)\} \qquad (3.13)$$

Also required are the series representations

$$V = V_o + \sum_{m=1}^{\infty} |A|^{2m} V_m(z), \qquad (3.14)$$

and

$$\frac{dA}{dt} = \pi^2 a_o A + A \sum_{m=1}^{\infty} a_m |A|^{2m} \qquad (3.15)$$

where V stands for Γ or Υ. (See reference *12* for a discussion of the validity of these representations.) With these series representations a consideration of the set of nonlinear partial differential equations (3.1) to (3.6) can be reduced to a consideration of Equation (3.15) called the *amplitude equation.* The problem then reduces to evaluating the coefficients a_o, a_m and studying Equation (3.15) which is a nonlinear ordinary differential equation. Thus, the amplitude function $A(t)$ replaces the exponential function of linear stability theory; in this way it is hoped that a solution which is valid for a greater range in time than linear theory can be generated. From the form of Equation (3.15) it is apparent that $\pi^2 a_o$ is the exponential time factor of linear theory. For, if higher order terms are negligible for infinitesimal amplitudes, the amplitude function $A(t)$ has the exponential time dependence $\exp(\pi^2 a_o)$.

Since much is known about nonlinear ordinary differential equations of the form (3.15) with a finite number of terms, the present approach reduces the original system of nonlinear partial differential equations to a tractable problem if Equation (3.15) is truncated. It is expected that for a range of amplitudes Equation (3.15) can be truncated as soon as a stabilizing term, that is, a term which produces a decrease in the magnitude of A (*7*), is encountered. Only if at least one stabilizing term is retained can the correct qualitative behavior of the system be predicted. Otherwise the amplitude of the instability would be a monotone increasing function of time, and behavior of this type is physically unrealistic if a cellular instability is to be expected. The modulus of the amplitude function $|A|$ as computed from the truncated amplitude equation may not remain small enough so that the series (3.15) is convergent. In this case the present asymptotic perturbation scheme makes sense only in those regions of parameter space where $|A|$ is such that the series representation (3.15) is convergent. Along these lines it is noteworthy that some recent experimental observations by Donnelly (*2*) on the growth of an instability in the related problem of viscous flow between rotating cylinders are represented to good approximation by Equation (3.15) truncated at $m = 1$.

PERTURBATION EQUATIONS

Substitution of the series representations (3.11) to (3.15) into the system of Equations (3.1) to (3.6) generates a complicated set of equations for the n^{th} Fourier component of the perturbation velocity, temperature, and concentration fields and also for the mean temperature and concentration fields. Entering into these equations is the modulus of the amplitude function $|A|$. By grouping terms in the equations with respect to powers of $|A|$, and by equating each coefficient of a power of $|A|$ to zero, one can obtain an infinite number of linear ordinary differential equations. The important characteristics of this infinite system of equations are their linearity and the property that the equations can be solved consecutively. The following systems of equations are characteristic of those obtained by the above process.

$0(|A|)$, $n = 1$:

$$(\pi^2 a_o + 2N_{Pr}(\alpha\pi)^2) w_{10}'' - (\alpha\pi)^2 (\pi^2 a_o + N_{Pr}(\alpha\pi)^2) w_{10} - N_{Pr} w_{10}^{iv} + (\alpha\pi)^2 (N_{Pr} T_{10} - N_{Sc} C_{10}) = 0,$$
$$(\pi^2 a_o + (\alpha\pi)^2) T_{10} - T_{10}'' + \Upsilon_o' w_{10} = 0,$$
$$(\pi^2 a_o + \eta(\alpha\pi)^2) C_{10} - \eta C_{10}'' + \Gamma_o' w_{10} = 0 \qquad (4.1)$$

$0(|A|)$, $n = 2$:

$$2(\pi^2 a_o + 4N_{Pr}(\alpha\pi)^2) w_{20}'' - 8(\alpha\pi)^2 (\pi^2 a_o + 2N_{Pr}(\alpha\pi)^2) w_{20} - N_{Pr} w_{20}^{iv} + 4(\alpha\pi)^2 (N_{Pr} T_{20} - N_{Sc} C_{20}) = w_{10}' w_{10}'' - w_{10} w_{10}''',$$
$$2(\pi^2 a_o + 2(\alpha\pi)^2) T_{20} - T_{20}'' + \Upsilon_o' w_{20} = \frac{1}{2}(w_{10}' T_{10} - w_{10} T_{10}'),$$
$$2(\pi^2 a_o + 2\eta(\alpha\pi)^2) C_{20} - \eta C_{20}'' + \Gamma_o' w_{20} = \frac{1}{2}(w_{10}' C_{10} - w_{10} C_{10}') \qquad (4.2)$$

Mean equations, $n = 1$

$$2\pi^2 a_o^{(r)} \Upsilon_1 - \Upsilon_1'' = -\frac{1}{2} Re(w_{10} T_{10})'$$

(Re = "real part of")

$$2\pi^2 a_o^{(r)} \Gamma_1 - \eta \Gamma_1'' = -\frac{1}{2} Re(w_{10} C_{10})' \qquad (4.3)$$

$0(|A|^3)$, $n = 1$:

$$\{\pi^2(a_o + 2a_o^{(r)}) + 2N_{Pr}(\alpha\pi)^2\} w_{11}'' - (\alpha\pi^2)\{\pi^2(a_o + 2a_o^{(r)}) + N_{Pr}(\alpha\pi)^2\} w_{11} + (\alpha\pi)^2 (N_{Pr} T_{11} - N_{Sc} C_{11}) - N_{Pr} w_{11}^{iv} = -a_1(w_{10}'' - (\alpha\pi)^2 w_{10}) + \frac{1}{2}(\alpha\pi)^2 \left\{\frac{3}{2}\tilde{w}_{10} w_{20}' + 3\tilde{w}_{10}' w_{20}\right\} + \frac{1}{2}\left\{\frac{1}{2} w_{20}' \tilde{w}_{10}'' + w_{20}\tilde{w}_{10}''' - \tilde{w}_{10}' w_{20}'' - \frac{1}{2}\tilde{w}_{10} w_{20}'''\right\},$$
$$\{\pi^2(a_o + 2a_o^{(r)}) + (\alpha\pi)^2\} T_{11} + \Upsilon_o' w_{11} - T_{11}'' = \frac{1}{2}\left\{-2\tilde{w}_{10}' T_{20} - \tilde{w}_{10} T_{20}' - \frac{1}{2} w_{20}' \tilde{T}_{10} + w_{20}\tilde{T}_{10}'\right\} - a_1 T_{10} - \Upsilon_1' w_{10},$$
$$\{\pi^2(a_o + 2a_o^{(r)}) + \eta(\alpha\pi)^2\} C_{11} + \Gamma_o' w_{11} - \eta C_{11}'' = \frac{1}{2}\left\{-2\tilde{w}_{10}' C_{20} - \tilde{w}_{10} C_{20}' - \frac{1}{2} w_{20}' \tilde{C}_{10} + w_{20}\tilde{C}_{10}'\right\} - a_1 C_{10} - \Gamma_1' w_{10} \qquad (4.4)$$

Here a prime denotes $\frac{d}{dz}$, and $a_o^{(r)}$ is the real part of a_o. The constant a_2 appears in the $(0(|A|^5), n = 1)$ system of equations which are similar in form to the systems of equations above; in order to compute a_2 the $(0(|A|), n = 3$ system of equations and the mean equations for $n = 2$ must also be solved. The equations above are to be solved subject to the following boundary conditions:

$$w_{io} = w_{io}'' = T_{io} = C_{io} = 0, \quad i = 1, 2, 3$$
$$w_{1j} = w_{1j}'' = C_{1j} = \Upsilon_j = \Gamma_j = T_{21} = C_{21} = w_{21} = w_{21}'' = 0, \quad j = 1, 2$$

METHOD OF DETERMINING THE COEFFICIENTS a_m

In order to determine the unknown coefficients a_m entering into the amplitude equation (3.15), one uses the observation that the system of equations obtained by the process described previously can be grouped into the general form

$$\mathcal{L} \cdot \mathbf{q} + k\mathcal{K} \cdot \mathbf{q} = \mathbf{F} \tag{5.1}$$

Here $\mathcal{L}$ is a linear dyadic operator which depends on a parameter $a_o^{(r)}$; that is, the real part of a_o, $\mathcal{K}$ is a linear dyadic operator, k is a known real valued constant, $\mathbf{F}$ is a known vector-valued function in which the constants a_m appear and $\mathbf{q}$ is a vector-valued dependent variable. Certain homogeneous boundary conditions complete the specification of the system. Let $\{\phi_n\}$ denote the set of eigenfunctions and $\{k_n\}$ the corresponding set of eigenvalues for the homogeneous equation, that is

$$\mathcal{L}_o \cdot \phi_n + k_n \mathcal{K} \cdot \phi_n = 0 \tag{5.2}$$

plus the homogeneous boundary conditions associated with the operator $\mathcal{L}$. Here $\mathcal{L}_o$ denotes the operator $\mathcal{L}$ evaluated at $a_o^{(r)} = 0$, and it is presumed the eigenvalues $\{k_n\}$ are simple. Let $\{\psi_n\}$ denote the set of eigenfunctions and $\{\lambda_n\}$ the corresponding set of eigenvalues for the homogeneous equation

$$\mathcal{L}_o^* \cdot \psi_n + \lambda_n \mathcal{K}^* \cdot \psi_n = 0 \tag{5.3}$$

plus the homogeneous boundary conditions adjoint to those associated with the operator $\mathcal{L}_o$. Here $\mathcal{L}_o^*$ and $\mathcal{K}^*$ are, respectively, the adjoint of the operators $\mathcal{L}_o$ and $\mathcal{K}$ and are defined in the following manner:

$$\begin{aligned} \int_o^1 \tilde{\mathbf{p}} \cdot (\mathcal{L}_o \cdot \mathbf{q}) dz &= \int_o^1 \mathbf{q} \cdot \widetilde{(\mathcal{L}_o^* \cdot \mathbf{p})} dz \quad , \\ \int_o^1 \tilde{\mathbf{p}} \cdot (\mathcal{K}_o \cdot \mathbf{q}) dz &= \int_o^1 \mathbf{q} \cdot \widetilde{(\mathcal{K}_o^* \cdot \mathbf{p})} dz \end{aligned} \tag{5.4}$$

where $\mathbf{q}$ satisfies the boundary conditions associated with $\mathcal{L}_o$. The adjoint boundary conditions, that is, those which $\mathbf{p}$ must satisfy, are determined by the requirement that relations (5.4) are fulfilled.

If $\{\phi_n\}$ and $\{\psi_n\}$ form complete sets (complete in a least square, or L_2, sense is sufficient here) in the domain and range of $\mathcal{L}_o$, then the following representations are valid:

$$\mathbf{q} = \sum_n b_n \phi_n \quad , \quad \mathbf{F} = \sum_n d_n \phi_n \tag{5.5}$$

where

$$b_n \equiv \int_o^1 \tilde{\psi}_n \cdot \mathbf{q}\ dz \quad , \quad d_n \equiv \int_o^1 \tilde{\psi}_n \cdot \mathbf{F}\ dz \tag{5.6}$$

(In the present problem the completeness of the sets $\{\phi_n\}$ and $\{\psi_n\}$ follows from the complete continuity of the operators $\mathcal{L}_o$ and $\mathcal{L}_o^*$.) Hence if

$$(\mathbf{r};\mathbf{s}) \equiv \int_o^1 \tilde{\mathbf{s}} \cdot \mathbf{r}\ dz \quad , \tag{5.7}$$

then from Equation (5.1) it follows that

$$(\mathcal{L} \cdot \mathbf{q};\ \psi_n) + k(\mathcal{K} \cdot \mathbf{q};\ \psi_n) = (\mathbf{F};\psi_n) \tag{5.8}$$

But

$$\begin{aligned} (\mathcal{L} \cdot \mathbf{q};\ \psi_n) &= (\mathbf{q};\ \mathcal{L}^* \cdot \psi_n) = -(\mathbf{q};\ \lambda_n \mathcal{K}^* \cdot \psi_n) + \\ & o(a_o^{(r)}) = -\tilde{\lambda}_n(\mathbf{q};\ \mathcal{K}^* \cdot \psi_n) + o(a_o^{(r)}) \end{aligned} \tag{5.9}$$

where $o(a_o^{(r)})$ is a term which tends to zero as $a_o^{(r)} \to 0$. Therefore,

$$(k - \lambda_n)\ (\mathbf{q};\ \mathcal{K}^* \cdot \psi_n) + o(a_o^{(r)}) = (\mathbf{F};\ \psi_n) \tag{5.10}$$

Also, if $\{\psi_n\}$ and $\{\phi_n\}$ are complete it follows that $\{k_n\} = \{\tilde{\lambda}_n\}$. Consequently, if $k = \tilde{\lambda}_n$ for $a_o^{(r)} = 0$, then from Equation (5.10) it follows that

$$(\mathbf{F};\ \psi_n)\big|_{a_o^{(r)}=0} = 0 \tag{5.11}$$

In order for the mathematical representation of the physical system to characterize meaningful physical behavior, Equation (5.11) must be satisfied. The coefficients a_m, $m = 1, 2, \ldots$, enter into the various $\mathbf{F}$'s generated by the systems of equation of the previous section and can be evaluated by requiring that Equation (5.11) be satisfied.

SOLUTION OF EQUATIONS NECESSARY TO DETERMINE a_0, a_1, and a_2

In the present analysis a stabilizing term is always encountered in the amplitude equation (3.15) truncated at $m = 2$. In order to determine the coefficients a_0, a_1, and a_2, only the equations up to $0(|A|^3)$ for n's up to 3 need be solved.

The solution to the $(0(|A|),\ n = 1)$ system of equations (4.1) represents the linear stability problem which has been considered elsewhere (6). Consequently, only the pertinent results are displayed here.

$$w_{10} = 2 \sin \pi z, \quad T_{10} = \pi^{-2} \tau_1 \sin \pi z, \quad C_{10} = \pi^{-2} \tau_3 \sin \pi z \tag{6.1}$$

where

$$\tau_1 = \frac{2R_t}{(a_o + \alpha^2 + 1)}, \quad \tau_3 = \frac{2R_c \eta^2}{\{a_o + \eta\ (\alpha^2 + 1)\}}, \qquad R_t = \pi^{-4} R_t^*, \quad R_c = \pi^{-4} R_c^* \tag{6.2}$$

and either $a_o^{(i)} = 0$ and

$$\begin{aligned} R_t = \alpha^{-2}(\alpha^2 + 1)\{&N_{Pr}^{-1} a_o^2 + (\alpha^2 + 1) \\ &(1 + N_{Pr}^{-1}) a_o + (\alpha^2 + 1)^2\} + \\ &\frac{\eta R_c \{a_o^2 + (\alpha^2 + 1)(1 + \eta) a_o + \eta\ (\alpha^2 + 1)^2\}}{\{a_o^2 + \eta\ (\alpha^2 + 1)\}^2} \end{aligned} \tag{6.3}$$

which characterizes a *stationary* instability or

$$a_o^{(i)2} = \frac{\alpha^2 \eta (1 - \eta) R_c}{\{2N_{Pr}^{-1} a_o^{(r)} + (\alpha^2 + 1)(1 + N_{Pr}^{-1})\}} - \{a_o^{(r)} + \eta\ (\alpha^2 + 1)\}^2 \tag{6.4}$$

and

$$\begin{aligned} R_t = \alpha^{-2}(\alpha^2 + 1)\ \{&N_{Pr}^{-1}(a_o^{(r)2} + a_o^{(i)2}) + \\ &(\alpha^2 + 1)(1 + N_{Pr}^{-1}) a_o^{(r)} + (\alpha^2 + 1)^2\} + \\ &\frac{\eta R_c\ \{a_o^{(r)2} + a_o^{(i)2} + (\alpha^2 + 1)(1 + \eta) a_o^{(r)} + \eta\ (\alpha^2 + 1)^2\}}{\{a_o^{(r)} + \eta\ (\alpha^2 + 1)\}^2 + a_o^{(i)2}} \end{aligned} \tag{6.5}$$

which characterizes an *oscillatory* instability. For a *marginal stationary instability*, that is $a_o^{(i)} \equiv 0$, $a_o^{(r)} \equiv 0$, Equation (6.3) simplifies to

$$R_t = \alpha^{-2}(\alpha^2 + 1)^3 + R_c \tag{6.6}$$

and for a marginal oscillatory instability, that is, $a_o^{(r)} \equiv 0$, Equations (6.4) and (6.5) simplify to

$$a_o^{(i)2} = \frac{\eta\ \alpha^2 (1 - \eta) R_c}{(1 + N_{Pr}^{-1})(\alpha^2 + 1)} - \eta^2 (\alpha^2 + 1)^2 \tag{6.7}$$

and

$$\begin{aligned} R_t = N_{Pr}^{-1}(1 + \eta)(\eta + N_{Pr})\alpha^{-2}(\alpha^2 + 1)^3 + \\ \eta\ (1 + N_{Pr})^{-1}(N_{Pr} + \eta) R_c \end{aligned} \tag{6.8}$$

Also it is not too difficult to establish that the *critical marginal stationary* as well as *oscillatory state* is located at $\alpha_{cr} = \sqrt{0.5}$, which is called the *critical wave number*. From an inspection of Equation (6.7) it is evident that to insure that $a_o^{(i)2} > 0$ it is necessary that $\eta < 1$ and

$$R_c > \frac{\eta\,(\alpha^2 + 1)^3\,(1 + N_{Pr})}{N_{Pr}\alpha^2(1 - \eta)} \equiv R_c^+ \qquad (6.9)$$

Also if it is presumed that the type of instability which manifests itself in a given marginal state of the system is that which occurs at the lower thermal Rayleigh number R_t, then an oscillatory instability occurs if $\eta < 1$, $R_c > R_c^+$ and $R_t > R_{t_{cr}}$. Here subscript *cr* denotes the critical marginal state at α_{cr}. Otherwise, a stationary instability occurs if $R_t > R_{t_{cr}}$.

$$a_2 = \frac{324 a_1 \{-4(1-\eta)a_1^2(R_c^+ - R_c) + \eta^{-2}(1-\eta^3)(R_c^{**} - R_c)\} - 4a_1^2\eta^{-1}(1-\eta)^2(R_c^+ - R_c)^2}{81\pi^2(1-\eta)(R_c^+ - R_c)} \qquad (7.4)$$

The solutions to the remaining equations which are necessary to determine the coefficients a_0, a_1, and a_2 are easy to obtain but cumbersome in structure; in the interest of brevity they are not displayed here.

COMPUTATION OF THE COEFFICIENTS a_0, a_1, and a_2

The magnitude of the coefficients a_o, a_1, and a_2 depends on the type of instability which is being considered. Two values of the coefficients, one appropriate for a stationary instability and one appropriate for an oscillatory instability, are computed.

In the case of a stationary instability Equation (6.3) leads to

$$a_o^3 + (\alpha^2 + 1)(N_{Pr} + \eta + 1)\,a_o^2 + \left\{(N_{Pr}\eta + N_{Pr} + \eta)(\alpha^2 + 1)^2 + \frac{N_{Pr}\alpha^2}{(\alpha^2 + 1)}(\eta R_c - R_t)\right\}a_o - N_{Pr}\eta\,\alpha^2\Delta R_t = 0 \qquad (7.1)$$

where

$$\Delta R_t \equiv R_t - 6.75 - R_c \qquad (7.2)$$

For given values of α, N_{Pr}, η, R_c and R_t the appropriate value of a_o can be computed from Equation (7.1). Similarly, Equations (6.4) and (6.5) are used to compute a_o for an oscillatory instability.

It is assumed here and hereafter that the wave number of the disturbance which propagates and the flow which is ultimately realized is α_{cr}, that is, the wave number of the first growing disturbance according to linear theory. Some recent results of Segel (7) seem to indicate the latter is a good assumption, at least for thermal Rayleigh numbers close enough to the critical value.

The method for evaluating the coefficients a_m which was outlined previously leads to the following results:

Stationary Instability, $\alpha = \alpha_{cr}$

$$a_1 = -\frac{(1 + \eta)}{2\eta}\,\frac{(R_c^* - R_c)}{(R_c^+ - R_c)}, \qquad (7.3)$$

[For $R_c \equiv 0$, $a_1 = \dfrac{-N_{Pr}}{2(1 + N_{Pr})}$ which agrees with the result obtained by Segel (7).]

Oscillatory Instability, $\alpha = \alpha_{cr}$

$$a_1^{(r)} \equiv 0, \qquad (7.5)$$

$$a_1^{(i)} = -6.75\,N_{Pr}^{-1}\eta^{-1}(1 - \eta)(1 + N_{Pr})^{-1}M^{-1}a_o^{(i)}\{\eta^{-2}(N_{Pr} + 1)(N_{Pr} + 2\eta + 1)\,R_c^+ - R_c\}, \qquad (7.6)$$

where

$$M = 4(1 - \eta^2)^2(1 + \eta)^{-2}(1 + N_{Pr})^{-2}(R_c^+ - R_c)^2 + 81(N_{Pr}\eta)^{-2}(1 + N_{Pr} + \eta)^2\,a_o^{(i)2}$$

The values of $a_2^{(r)}$ and $a_2^{(i)}$ are nonzero but the analytical expressions for these two constants are lengthy and are not presented here. Figure 1 displays the values of $a_2^{(r)}$ for values of the physical parameters characteristic of gaseous (*dotted line*) and liquid metal (*solid line*) systems.

By inspection of the explicit forms of the coefficients a_1 and a_2 for a stationary instability [Equations (7.3) and (7.4)], it is evident that for $\eta < 1$

$$a_1 \to +\infty \quad , \quad a_2 \to -\infty \qquad (7.7)$$

as $R_c \to R_c^+ -$. It is also easy to show that for $\eta < 1$

$$a_1 \to -\infty \quad , \quad a_2 \to +\infty \qquad (7.8)$$

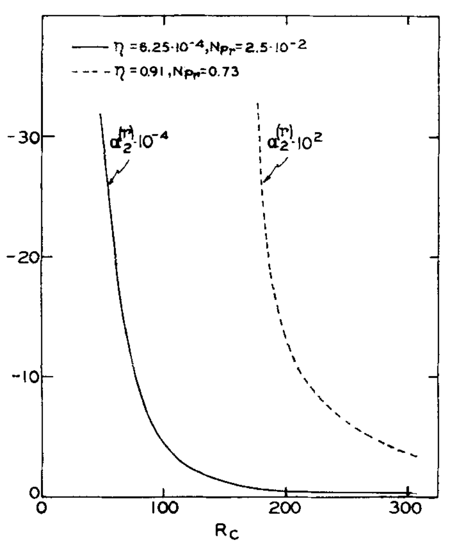

Fig. 1. Constants for amplitude equation, oscillatory instability.

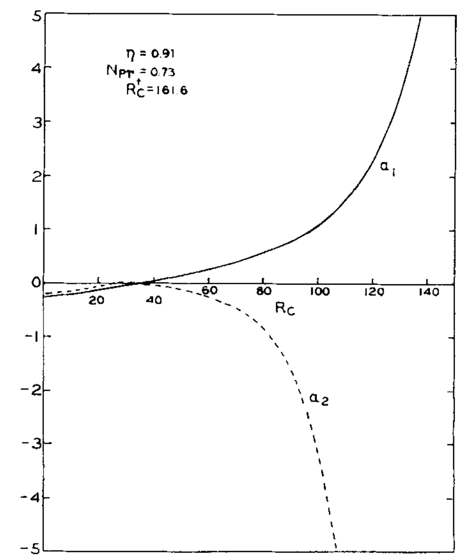

Fig. 2. Constants for amplitude equation, stationary instability.

as $R_c \rightarrow R_c^+ +$. Hence, there is an infinite discontinuity in the coefficients a_1 and a_2 for a stationary instability at the point $R_c = R_c^+$. The behavior of a typical gaseous system with $\eta = .91$ and $R_c^+ = 161.6$ is displayed in Figure 2.

For a stationary instability the coefficients a_o, a_1, and a_2 depend on the parameter η, which is the ratio of the Prandtl number to the Schmidt number (or mass diffusivity to thermal diffusivity). If $\eta < 1$, the coefficients a_1 and a_2 both pass through zero at least once and if $\eta > 1$ both coefficients are negative-definite functions of R_c. These properties greatly affect the behavior of solutions of the amplitude equation (3.15).

AMPLITUDE EQUATION

In all cases considered here the amplitude equation (3.15) is truncated at fifth-order, that is

$$\frac{dA}{dt} = \pi^2 a_o A + a_1 A|A|^2 + a_2 A|A|^4 \qquad (8.1)$$

This truncation is sound if $|A|^2$ remains sufficiently small for all time but is not necessarily sound if, for example, $|A|^2 = 0(1)$. Multiplying Equation (8.1) by $\widetilde{A}$ and the complex conjugate of Equation (8.1) by A, and then adding the resulting equation leads to

$$\frac{1}{2}\frac{d}{dt}|A|^2 = \pi^2 a_o^{(r)}|A|^2 + a_1^{(r)}|A|^4 + a_2^{(r)}|A|^6 \qquad (8.2)$$

Consequently, it is the real part of the coefficients a_m which determines the behavior of the magnitude of the amplitude of the disturbance. A coefficient $a_m^{(r)}$ is called stabilizing if it causes a decrease in the magnitude of the amplitude and destabilizing if it causes an increase. It is apparent from an inspection of Figures 1 and 2 that at least one of the coefficients, $a_1^{(r)}$ or $a_2^{(r)}$, is stabilizing (except at $R_c = R_c^*$). Therefore, in most cases the initial exponential behavior of the disturbances is modified and either finite amplitude dynamic equilibrium states are eventually attained or the disturbances decay and the system is stable. This behavior is also expected from physical considerations.

In the present case consideration of the amplitude A as obtained from Equation (8.1) can be restricted to the following five cases:

1. $0 \leq R_c \leq R_c^{++}$, $\eta < 1$; $a_1 < 0$, $a_2 < 0$
2. $R_c^{++} < R_c < R_c^*$, $\eta < 1$; $a_1 < 0$, $a_2 > 0$
3. $R_c^* < R_c \leq R_c^+$, $\eta < 1$; $a_1 > 0$, $a_2 < 0$

(cases 1–3: stationary instability)

4. $R_c^+ < R_c$, $\eta < 1$ — oscillatory instability
5. All R_c, $\eta > 1$; $a_1 > 0$, $a_2 > 0$ — stationary instability

where R_c^{++} is the value of R_c at the first zero of a_2. Once having prescribed the value of $R_c > 0$, that is, the unitless concentration difference across the layer, and η, the reciprocal of the Lewis number, the only parameter in the amplitude Equation (8.1) which can be varied is a_o whose magnitude and phase are determined by $(R_t - R_{t_{cr}})$.

Although Equation (8.2) can be solved exactly it is more revealing to investigate first the location and stability of its equilibrium points. At an equilibrium point $\frac{d}{dt}|A|^2 = 0$ and the modulus of the equilibrium amplitude A_e is characterized by the following expression:

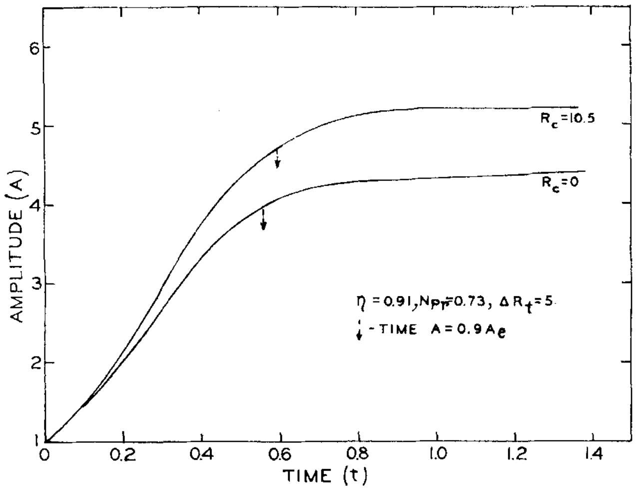

Fig. 3. Instantaneous amplitude, stationary instability.

$$|A_e|^2 = \frac{-a_1^{(r)2} \pm \sqrt{a_1^{(r)2} - 4\pi^2 a_o^{(r)} a_2^{(r)}}}{2a_2^{(r)}} \qquad (8.3)$$

Now the properties of $|A_e|$ are investigated in each of the five distinct R_c regions.

1. Stationary instability,
$0 < R_c < R_c^{++}$, $\eta < 0$; $a_1 < 0$, $a_2 < 1$

In order to insure that A_e^2 (for a stationary instability $A_e^2 = |A_e|^2$ and $a_m^{(r)} = a_m$) is positive, that is, the equilibrium amplitude be real, it is required that $a_o > 0$. The latter requirement is fulfilled only if $(R_t - R_{t_{cr}}) > 0$; that is, only a supercritical instability can occur. By inspection of Equation (8.3) it is apparent that only one equilibrium point occurs in this case, and a linear stability analysis establishes that it is a stable node. The attainment of the equilibrium amplitude for one case is displayed in Figure 3. The shape of the curves is similar to those determined experimentally by Donnelly (2) in a related system. Note that increasing the unitless concentration difference across the layer R_c increases the time necessary to attain 0.9 A_e. This behavior is to be expected, because the concentration field exerts a stabilizing influence in this particular R_c region.

2. Stationary instability
$R_c^{++} < R_c < R_c^*$, $\eta < 1$; $a_1 < 0$, $a_2 > 0$,

Here the coefficients a_1 and a_2 are different in sign and the behavior of the equilibrium amplitude A_e as a function of R_c differs markedly from the preceding R_c region. Here, for instance, an equilibrium amplitude can be attained both for $a_o > 0$, a supercritical instability and $a_o < 0$, a subcritical instability. The possibility of allowing both positive and negative values of a_o becomes apparent on inspection of Equation (8.3) with $a_1 < 0$ and $a_2 > 0$.

According to Equation (8.3) with $a_o > 0$ it is also required that

$$a_1^2 - 4\pi^2 a_o a_2 \geq 0 \qquad (8.4)$$

in order to insure that A_e is real. Consequently, according to the truncated amplitude equation there exists a maximum value of ΔR_t for which an equilibrium amplitude can be attained. (More terms must be retained in the amplitude equation if larger values of ΔR_t are to be investigated.)

In the present R_c region there are two equilibrium amplitudes, A_{e_1} and $A_{e_2} > A_{e_1}$, if $a_o > 0$. The locus of amplitudes A_{e_1} and A_{e_2} branch off from the separatrix locus

$$A_e = \sqrt{\frac{-a_1}{2a_2}}$$

which is obtained by substituting the equality appearing in relation (8.4) into Equation (8.3). Here again it is seen that the truncated amplitude equation (8.1) may not be valid for $A_e^2 = 0(1)$; however, A_e^2 of $0(1)$ is displayed in Figure 4 which illustrates the behavior alluded to above for the case of a typical gaseous system with $\eta < 1$. A linear stability analysis about each equilibrium amplitude establishes that A_{e_1} is a stable node and A_{e_2} is an unstable node. Consequently, if the truncated amplitude equation is valid, a disturbance with an amplitude less than A_{e_2} equilibrates to A_{e_1} at large times. The truncated amplitude Equation (8.1) is inappropriate for a disturbance with amplitude greater than A_{e_2} and more terms must be retained in the amplitude equation in order to consider such disturbances.

As previously mentioned an equilibrium amplitude A_{e_3} can be realized at a subcritical state and $A_{e_3} > A_{e_2}$ follows from Equation (8.3). In this case a linear stability analysis establishes that this equilibrium point is an unstable node. Consequently, in regions where the truncated amplitude equation is valid a disturbance whose magnitude is less than A_{e_3} equilibrates to zero amplitude at large time; the truncated amplitude equation is inappropriate for disturbances with amplitudes greater than A_{e_3}. It should be noted that a subcritical instability is a purely nonlinear effect; that is, according to linear theory the system is stable to all disturbances if $R_t - R_{t_{cr}} < 0$.

3. Stationary instability
$R_c^* < R_c < R_c^+$, $\eta < 1$; $a_1 > 0$, $a_2 < 0$

Here the coefficients a_1 and a_2 again differ in sign, and either a subcritical or supercritical instability can occur. However, because $a_1 > 0$ and $a_2 < 0$, a subcritical instability leads to two equilibrium amplitudes A_{e_1} and $A_{e_2} \geqq A_{e_1}$, and a supercritical instability to one equilibrium amplitude A_{e_3}. A linear stability analysis about each equilibrium amplitude establishes that A_{e_1} is an unstable node and A_{e_2} and A_{e_3} are stable nodes. Therefore, if a disturbance in a subcritical state of the system initially exceeds an amplitude A_{e_1}, it equilibrates to an equilibrium amplitude A_{e_2} at large time; whereas, if a disturbance has an amplitude less than A_{e_1} it decays. This type of behavior is illustrated in Figure 4. [Note here and hereafter that the truncated amplitude Equation (8.1) may not be sound for $|A_e|^2 = 0(1)$.]

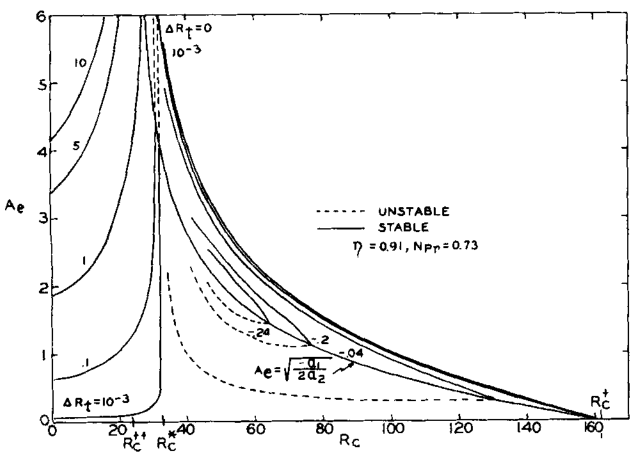

Fig. 4. Equilibrium amplitude, $\eta = 0.91$, $N_{pr} = 0.73$.

Although a supercritical instability can occur here a subcritical type of instability is most likely to be encountered experimentally. In experiments it is most likely that the magnitude of η, N_{Pr}, and R_c (or R_t) would be held relatively constant and the value of R_t (or R_c) increased until an instability is detected. Consequently, if there are sufficiently large disturbances present a subcritical type of instability should determine the experimentally determined value of the critical thermal Rayleigh number R_t. The dotted lines in Figure 4 indicate what is meant by sufficiently large disturbances [according to the truncated amplitude Equation (8.1)] for $\eta = 0.91$ and $N_{Pr} = 0.73$. In Figure 4 it is apparent that the initial amplitude of the disturbance need not be very large if $|(R_t - R_{t_{cr}})|$ is small. Indeed, if $R_t = R_{t_{cr}}$, any disturbance is sufficient.

In order to insure that A_e is real it is necessary that the radical in Equation (8.3) be real. The limiting condition is

$$a_1^2 - 4\pi^2 a_o a_2 = 0 \tag{8.5}$$

and the corresponding equilibrium amplitude is

$$A_e = \sqrt{\frac{-a_1}{2a_2}} \tag{8.6}$$

The locus of equilibrium amplitudes (8.6) is evident as the separatrix between the locus of stable equilibrium amplitudes A_{e_2} and the unstable ones A_{e_1} in Figure 4. Also the restriction imposed by Equation (8.5) appears to represent a realistic physical phenomenon, since a limiting decay rate for which the nonlinear effects could give rise to an instability is to be expected. However, because of the smallness of the difference $(R_t - R_{t_{cr}})$ associated with the limiting decay rate, it appears that subcritical instabilities would be difficult to detect experimentally in this or a similar system. Although the system being treated in this analysis is highly idealized, it is felt that the same qualitative behavior is characteristic of more realistic systems.

As previously mentioned

$$a_1 \to \infty \quad \text{and} \quad a_2 \to -\infty$$

as $R_c \to R_c^+$. The point $R_c = R_c^+$ is also significant, because according to linear theory a stationary instability occurs initially for $R_c < R_c^+$ and an oscillatory instability for $R_c > R_c^+$. With the use of Equations (7.3) and (7.4) it follows that

$$\frac{a_1}{a_2} \to 0$$

as $R_c \to R_c^+ -$. Coupling the latter result with Equation (8.3) establishes that $A_e \to 0$ as $R_c \to R_c^+-$ [in regions where the truncated amplitude Equation (8.1) is valid]. This result is also reasonable from a physical standpoint and tends to lend credence to the validity of Equation (8.3) in the vicinity of R_c^+. Thus, if the physical system initially experiences a stationary instability in an R_c stationary region and if external conditions are then gradually adjusted until R_c enters the R_c oscillatory region, the amplitude of the stationary instability would decrease, become zero at $R_c = R_c^+$, and an oscillatory instability would appear for $R_c > R_c^+$. To the order of the present approximation the system experiences a metastable state at $R_c = R_c^+$ even though a growing disturbance can occur according to linear theory.

In regard to a subcritical instability it is interesting to note that Equation (7.1) for a_o has three real negative roots if $R_c^* < R_c < R_c^+$, $\eta < 1$. However, the only one which can lead to a real equilibrium amplitude A_e is the largest one, that is the slowest decaying disturbance.

4. Oscillatory instability,

$$R_c > R_c^+, \ \eta < 1$$

An oscillatory instability occurs in this R_c region, and the equilibrium corresponds to an undamped oscillation. Since it was pointed out previously that $a_1^{(r)} = 0$, the modulus of the equilibrium amplitude (8.3) becomes

$$|A_e| = \left(\frac{-\pi^2 a_o^{(r)}}{a_2^{(r)}} \right)^{1/4} \tag{8.7}$$

and the corresponding equilibrium wave velocity is

$$\omega_e = \pi^2 a_o^{(i)} + a_1^{(i)}|A_e|^2 + a_2^{(i)}|A_e|^4 \tag{8.8}$$

Here computations were made for three characteristic systems. The systems and results are as follows:

1. Liquids:

$$\eta = 0.01, \ N_{Pr} = 8; \ a_2^{(r)} > 0$$

2. Gases:

$$\eta = 0.91, \ N_{Pr} = 0.73; \ a_2^{(r)} < 0 \tag{8.9}$$

3. Liquid metals:

$$\eta = 6.25 \cdot 10^{-4}, \ N_{Pr} = 2.5 \cdot 10^{-2}; \ a_2^{(r)} < 0$$

From an inspection of Equation (8.7) coupled with the results (8.9), it is apparent that a subcritical oscillatory instability can occur in the liquid system. There appears to be a locus of $\eta < 1$ and N_{Pr} values which separate the regions of subcritical and supercritical oscillatory instabilities; however, this has not been investigated. The computations which were performed do establish that the equilibrium amplitudes are much smaller than those associated with a corresponding stationary instability and that $\omega_e \approx \pi^2 a_o^{(i)}$. The occurrence of very small amplitudes and $\omega_e \approx \pi^2 a_o^{(i)}$ are in complete accord with experimental observations of Nakagawa and Frenzen (*4*) in a rotating layer of mercury heated from below. The present results and the observations of Nakagawa and Frenzen indicate that the wave velocity of a growing disturbance according to linear theory is not distorted to any great extent by the nonlinear effects which can have such a large effect on the magnitude of the amplitude.

5. Stationary instability,

$$0 \leqq R_c, \ \eta \geqq 1; \ a_1 < 0, \ a_2 < 0$$

Here an oscillatory instability cannot occur, and the behavior predicted by the truncated amplitude Equation (8.1) differs greatly from that corresponding to a system with $\eta < 1$. In this R_c region both a_1 and a_2 are decreasing negative-definite functions of R_c; consequently, a subcritical instability cannot occur. In this case, that is, $0 \leqq R_c$, $\eta \geqq 1$, the equilibrium amplitude A_e is a monotone decreasing function of R_c.

The latter behavior is in sharp contrast to the behavior for a system with $R_c < R_c^{++}$, $\eta < 1$ in which the equilibrium amplitude is a monotone increasing function of R_c. Since the concentration difference tends to stabilize the initial potentially unstable system, the behavior exhibited by $\eta > 1$ systems is expected on a intuitive bases. However, this intuitive expectation is violated by $\eta < 1$ systems.

CONVECTIVE TRANSPORT

The convective transport of heat and mass is important from an applied as well as a theoretical point of view. Its importance in applications is obvious. Its theoretical importance stems from the observation that the condition of maximum convective transport appears to be the criterion which eliminates the indeterminacy in the cellular shape of the instability which has the most tendency to develop (*3*). Therefore, it is important to determine in what manner the various physical parameters effect the convective transport. It has also been established both theoretically (*11*) and experimentally (*4*) that in a rotating fluid layer heated from beneath the convective transport is much smaller for an oscillatory instability than for a stationary instability. The question of the validity of this property here arises, and its answer may provide a clearer insight into the constraining effect alluded to by Veronis (*11*).

The average vertical convective transport in a dynamic equilibrium state of the present system is

$$\overline{\left(\int_o^1 w\psi \, dz \right)} = \frac{1}{2} \sum_{n=1}^{\infty} |A_e|^{2n} Re \left[\int_o^1 \left\{ w_{no} \widetilde{\psi}_{no} + \sum_{m=1}^{\infty} (w_{no} \widetilde{\psi}_{nm} + w_{nm} \widetilde{\psi}_{no}) |A_e|^{2m} + \sum_{m=1}^{\infty} \sum_{p=1}^{m-1} w_{np} \widetilde{\psi}_{nm-p} |A_e|^{2m} \right\} dz \right] \tag{9.1}$$

Here ψ stands for T or C.

To $0(|A_e|^4)$ the convective transport of heat h and the convective transport of mass m at an equilibrium state are

$$h = R_t + \frac{1}{2} Re(\pi \tau_1 - E|A_e|^2) |A_e|^2 \tag{9.2}$$

$$m = \eta^2 R_c + \frac{1}{2} Re(\pi \tau_3 - F|A_e|^2) |A_e|^2 \tag{9.3}$$

where

$$E \equiv \frac{(2 \tau_1 a_1 + \tau_1^{(r)})}{2\{(\alpha^2 + 1) + i a_o^{(i)}\}},$$

$$F \equiv \frac{(2\eta \tau_3 a_1 + \tau_3^{(r)})}{2\eta\{\eta(\alpha^2 + 1) + a_o^{(i)}\}}$$

In particular, for a stationary instability Equations (9.2) and (9.3) take the form

$$\frac{h}{R_t^*} = 1 + \frac{A_e^2}{\pi^2(\alpha^2+1)} \left\{ 1 - \frac{A_e^2}{2\pi^2(\alpha^2+1)} \left[1 - \frac{(1+\eta)}{\eta} \frac{(R_c^* - R_c)}{(R_c^+ - R_c)} \right] \right\}, \tag{9.4}$$

$$\frac{m}{\eta^2 R_c^*} = 1 + \frac{A_e^2}{\eta\pi^2(\alpha^2+1)} \left\{ 1 - \frac{A_e^2}{2\eta^2\pi^2(\alpha^2+1)} \left[1 - (1+\eta) \frac{(R_c^* - R_c)}{(R_c^+ - R_c)} \right] \right\} \tag{9.5}$$

For a system subjected to only heat transfer the average vertical heat transfer to $0(|A_e|^4)$, as computed from Equation (9.4), is identical to the $0(|A_e|^4)$ value previously computed by Malkus and Veronis (*3*). The transport of heat and mass to $0(|A_e|^6)$ was also computed here but in the interest of brevity is not specifically displayed.

Since powers of A_e occur in Equation (9.1), the average vertical transport of heat and mass appears to offer a criterion for estimating the range of validity of the truncated amplitude equation. Figure 5 displays the average vertical transport of heat to $0(|A_e|^4)$ for an $\eta < 1$ system in the range $0 \leqq R_c \leqq R_c^*$. (The behavior of the average vertical transport of mass is similar.) The dotted portion indicates the $R_t/R_{t_{cr}}$ range of questionable validity. Note

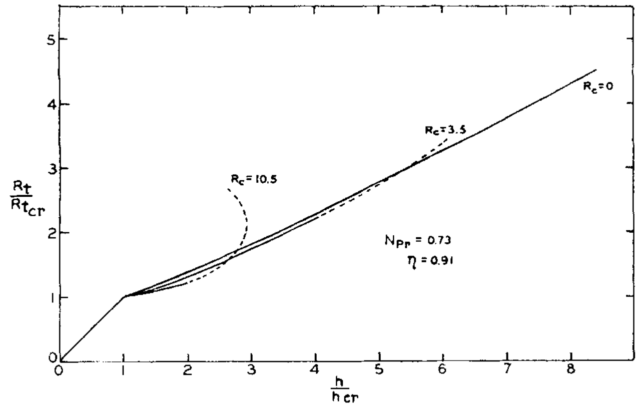

Fig. 5. Average vertical heat transport, η = 0.91, N_{pr} = 0.73.

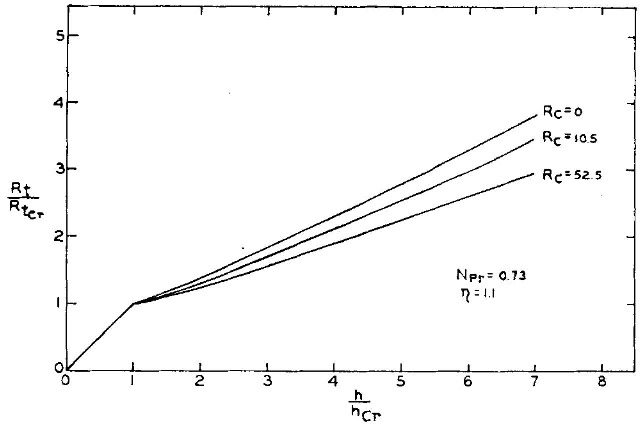

Fig. 7. Average vertical heat transport, η = 1.1, N_{pr} = 0.73.

that as R_c approaches R_c^* the range of validity appears to decrease; this type of behavior is to be expected, since the equilibrium amplitude grows very rapidly as $R_c \to R_c^*-$.

Figure 6 displays the average vertical transport of heat to $0(|A_e|^6)$ for a state of the system which exhibits a subcritical instability. Note that there are three equilibrium states: stable, metastable, and unstable. For the state of the system displayed in Figure 6 the initial quiescent system is in a stable equilibrium state if $R_t < 54.50$ and in a metastable equilibrium state if $54.50 \leqq R_t \leqq 59.75$. In a stable equilibrium state the system is stable to all roll cell disturbances; in a metastable state the system is stable only to certain roll cell disturbances. For instance, if a disturbance causes the average vertical heat transport to increase to values greater than the dotted portion of Figure 6 then the system is unstable; otherwise it is stable. In contrast, the average vertical transport of heat corresponding to the dotted portion of Figure 6 can never be maintained, because it corresponds to an unstable equilibrium state. Consequently, a plot of average vertical transport of heat vs. R_t for a R_c subcritical region always possesses a discontinuity. The average vertical transport of mass behaves in the same manner.

Figure 7 displays the average vertical transport of heat for an $\eta > 1$ system. (The average vertical transport of mass is similar.) Here the truncated amplitude equation yields results which appear to be valid for a large range of $R_t/R_{t_{cr}}$ and R_c values. An apparently good representation results because the equilibrium amplitude for an $\eta > 1$ system is a monotone decreasing function of R_c.

For an oscillatory instability the average vertical transport of heat and mass to $0(|A_e|^2)$ is

$$\frac{h}{R_t^*} = 1 + \frac{(\alpha^2 + 1)\,|A_e|^2}{\pi^2\{(\alpha^2 + 1)^2 + a_o^{(1)2}\}} \qquad (8.10)$$

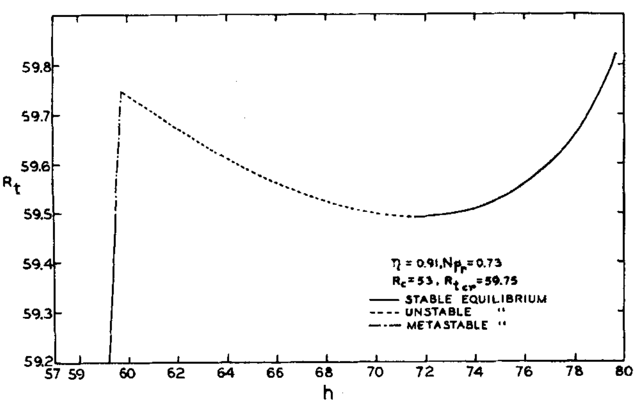

Fig. 6. Average vertical heat transport in subcritical R_c region.

$$\frac{m}{\eta^2 R_c^*} = 1 + \frac{\eta\,(\alpha^2 + 1)\,|A_e|^2}{\pi^2\,\{\eta^2(\alpha^2 + 1)^2 + a^{(1)2}\}} \qquad (8.11)$$

These expressions are expected to be good approximations, because the amplitudes of the oscillatory regime are very small. In particular, for a system in which $R_c = 166$, $\eta = 0.91$, $N_{Pr} = 0.73$, $\Delta R_t = 10^{-3}$, and $\alpha = \sqrt{.5}$, Equation (8.10) leads to

$$\frac{h}{R_t^*} = 1 + 1.37 \cdot 10^{-6}$$

Note how little the convective transport associated with an oscillatory instability contributes; the convective transport associated with a stationary instability in the present system is over a thousand times larger. This significant difference has also been observed experimentally in a related system (*4*).

SUMMARY

In order to place the results obtained here in their proper setting it is necessary to recapitulate the finite amplitude effects which are and are not considered. The analysis accounts for the distortion of the mean temperature and concentration fields and the configuration of the local temperature and concentration fields to $0(|A|^4)$. However, the prediction of the cellular pattern with the greatest tendency to form in the system and the tendency for a change in cell size are not considered in the present analysis; these latter items are treated in a sequel to this work.

In the present analysis the Stuart-Watson perturbation method leads to a valid mathematical characterization of the system in certain regions of parameter space. The method is easily extended to other systems, to other forms of disturbances, and also to the prediction of the cellular pattern with the greatest tendency to develop.

It is found that in order to obtain valid qualitative and quantitative information a higher order of approximation is necessary for a system which is capable of an oscillatory instability than one which is not. If the present system is operated in a region in parameter space where an oscillatory instability is possible, then both subcritical and supercritical stationary instabilities can occur. In this case it appears that many terms must be retained in the amplitude equation in order to obtain quantitative information if $(R_t - R_{t_{cr}})$ is reasonably large. In contrast, the Stuart-Watson method converges rapidly for the present system when the system is not capable of exhibiting an oscillatory instability. This analysis clearly illustrates the complexity in the behavior of a system capable of an oscillatory instability. Also it is noteworthy that here the amplitude of an existing stationary instability tends to zero before an oscillatory instability develops.

In many engineering applications the flux of heat and mass is of primary interest and the perturbation method used here offers a method for computing these fluxes from first principles. The magnitude of the flux of heat and mass to a great extent depends on the type of instability, namely, oscillatory or stationary. Thus, it is important to be able to determine the regions in parameter space where the two types of instabilities can occur. In the present case it is sufficient to consider only the linear stability problem in order to locate these regions; however, the linear stability problem may not be sufficient for other systems.

ACKNOWLEDGMENT

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NOTATION

A = amplitude function
$|A|$ = modulus of A
a_m = coefficients appearing in Equation (3.15)
C = dimensionless concentration perturbation
D = specie diffusion coefficient
E, F = constants defined with Equations (9.2) and (9.3)
F, **q**, **p**, **r**, **s** = vector-valued functions
g = gravitational field
h = total average vertical heat transport
i = $\sqrt{-1}$
$\{k_n\}$ = set of eigenvalues appearing in Equation (5.3)
$\mathcal{L}, \mathcal{L}_o, \mathcal{L}_o^*, \mathcal{K}, \mathcal{K}^*$ = linear dyadic operators
l = depth of fluid layer
m = total average vertical mass transport
N_{Pr} = Prandtl number = $\dfrac{\nu}{\kappa}$
N_{Sc} = Schmidt number = $\dfrac{\nu}{D}$
p^* = pressure
R_c^* = mass Rayleigh number [see Equation (3.9)] = $\pi^4 R_c$
R_c^* = $\dfrac{6.75\,\eta^2}{(1-\eta^2)}$
R_c^{**} = $\dfrac{\eta(1-\eta^2)R_c^*}{(1-\eta^3)}$
R_c^+ = $\dfrac{\eta(\alpha^2+1)^3(1+N_{Pr})}{N_{Pr}\alpha^2(1-\eta)}$
R_c^{++} = magnitude of R_c at first zero of a_2 (stationary instability)
Re = real part of
R_t^* = thermal Rayleigh number [see Equation (2.4)] = $\pi^4 R_t$
T = dimensionless temperature perturbation
t = dimensionless time
(x, y, z) = dimensionless independent variables
(u, v, w) = dimensionless velocity vector
$\times$ = vector multiplication
$\cdot$ = scalar multiplication
∇^2 = $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}$

Greek Letters

α = dimensionless wave number
β = species coefficient of expansion
γ = thermal coefficient of expansion
η = N_{Pr}/N_{Sc}
κ = thermal diffusivity
θ = dimensionless temperature = $T + T = \theta^*/\theta.$
$\{\lambda_m\}$ = set of eigenvalues appearing in Equation (5.3)
ν = kinematic viscosity
ρ = density
$\Delta\rho_t$ = $-\rho_r\gamma(\theta_B^* - \theta_T^*)$
$\Delta\rho_c$ = $\rho_r\beta(\zeta_B^* - \zeta_T^*)$
τ_1, τ_2 = constants defined by Equation (6.2)
$\{\phi_n\}$ = set of eigenfunctions appearing in Equation (5.2)
$\{\psi_n\}$ = set of eigenfunctions appearing in Equation (5.3)
ω = wave velocity of oscillatory instability
Γ = dimensionless mean concentration
Λ = dimensionless concentration = $\Gamma + C = \Lambda^*/\Lambda.$
Υ = dimensionless mean temperature

Subscripts

B = bottom surface
cr = critical state
e = equilibrium
r = reference state
z = $\dfrac{\partial}{\partial z}$
t = $\dfrac{\partial}{\partial t}$
T = top surface

Superscripts

(r) = real part
(i) = imaginary part
$'$ = $\dfrac{d}{dz}$

Overhead

$\sim$ = complex conjugate
$-$ = x-spatial average

Measure numbers

$t.$ = $\dfrac{l^2}{\kappa}$
$u.$ = $\dfrac{\kappa}{l}$
$x.$ = l
$\Lambda.$ = $\dfrac{D^2 N_{Sc}}{g|\beta|l^3}$
$\theta.$ = $\dfrac{\kappa^2 N_{Pr}}{|\gamma|gl^3}$

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